Name:…………………………………………………………………………………….

Student Number: ……………………………………………………………………..

Test 5 on WPPH16001.2018-2019
"Electricity and Magnetism"

Content: 10 pages (including this cover page)

Friday May 24 2019; A. Jacobshal 01, 9:00-11:00

- Write your full name and student number in the place above
- Write your answers in the designated areas
- Read the questions carefully
- Compose your answers is such a way that it is well indicated which (sub)question they address
- Reversed sides of each page are left blank intentionally and could be used for draft answers
- Do not use a red pen (it's used for grading) or a pencil
- Books, notes, phones, tablets, smartwatches and headphones are not allowed. Calculators and dictionaries are allowed.

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

For administrative purposes; do NOT fill the table

The weighting of the questions:

	Maximum points	Points scored
Question 1	15	
Question 2	15	
Question 3	15	
Question 4	5	
Total	**50**	

Grade = 1 + 9 x (score/max score).

Grade: _________

Question 1. (15 points)
An infinitely long cylindrical tube, of radius *a,* moves at constant speed *v* along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius *b,* is another cylinder, moving with the same velocity but carrying the opposite charge $(-\lambda)$.

1. Show that electric field is zero for $s < a$ and $s > b$, while between the cylinders: $\vec{\mathbf{E}} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}$

(2.5 points)

2. Show that magnetic field is zero for $s < a$ and $s > b$, while between the cylinders: $\vec{\mathbf{B}} = \frac{\mu_0}{2\pi}\frac{\lambda v}{s}\hat{\boldsymbol{\phi}}$

(2.5 points)

3. Find the energy per unit length W/ℓ stored in the fields. (5 points)
4. Find the momentum per unit length $\vec{\mathbf{p}}/\ell$ in the fields. (5 points)

Answers to Question 1 (Problem 8.14a,b) (10 points)

1. Because of symmetry, $\vec{\mathbf{E}}$ is directed radially (along $\hat{\mathbf{s}}$). Using a cylindrical Gaussian curface of a radius $a < s < b$ and length l:

$$\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{1}{\epsilon_o} Q_{encl}; E \cdot 2\pi s l = \frac{1}{\epsilon_o} \lambda l;\ \vec{\mathbf{E}} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{s} \hat{\mathbf{s}}$$

For $s < a$ and $s > b$, $\vec{\mathbf{E}} = 0$ because $Q_{encl} = 0$ (2.5 points)

2. Because of symmetry, $\vec{\mathbf{B}}$ is directed curcularily (along $\hat{\boldsymbol{\phi}}$). Using a curcular Amperian loop of a radius $a < s < b$:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{encl}; B \cdot 2\pi s = \mu_0 \cdot \lambda v;\ \vec{\mathbf{B}} = \frac{\mu_0}{2\pi} \frac{\lambda v}{s} \hat{\boldsymbol{\phi}}$$

For $s < a$ and $s > b$, $\vec{\mathbf{B}} = 0$ because $I_{encl} = 0$ (2.5 points)

3. (5 points)

The energy density:

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = \frac{1}{2}\left[\epsilon_0 \left(\frac{1}{2\pi\epsilon_0}\right)^2 \frac{\lambda^2}{s^2} + \frac{1}{\mu_0}\left(\frac{\mu_0}{2\pi}\right)^2 \frac{\lambda^2 v^2}{s^2}\right] = \frac{\lambda^2}{8\pi^2\epsilon_0}(1 + \epsilon_0\mu_0 v^2)\frac{1}{s^2}$$

$$\frac{W}{\ell} = \frac{\lambda^2}{8\pi^2\epsilon_0}(1 + \epsilon_0\mu_0 v^2)\frac{1}{\ell}\int_0^\ell dl \int_0^{2\pi} \int_a^b \frac{1}{s^2}\, s ds d\varphi = \frac{\lambda^2}{4\pi\epsilon_0}(1 + \epsilon_0\mu_0 v^2) \ln\left(\frac{b}{a}\right)$$

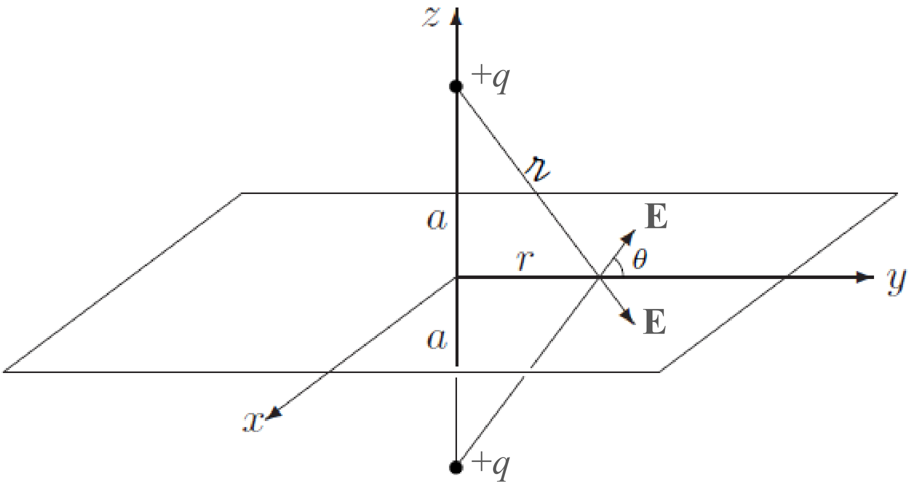
4. (5 points)

The momentum density:

$$\vec{\mathbf{g}} = \epsilon_0(\vec{\mathbf{E}} \times \vec{\mathbf{B}}) = \epsilon_0 \left(\frac{1}{2\pi\epsilon_0} \frac{\lambda}{s}\right)\left(\frac{\mu_0}{2\pi} \frac{\lambda v}{s}\right)\hat{\mathbf{z}} = \frac{\mu_0 \lambda^2 v}{4\pi^2 s^2}\ \hat{\mathbf{z}}$$

$$\frac{\vec{\mathbf{p}}}{\ell} = \frac{\mu_0 \lambda^2 v}{4\pi^2}\ \hat{\mathbf{z}} \int_a^b \frac{1}{s^2}\ 2\pi s ds = \frac{\mu_0 \lambda^2 v}{2\pi} \ln\left(\frac{b}{a}\right)\ \hat{\mathbf{z}}$$

Question 2 (15 points)



Consider two equal point charges $+q$, separated by a distance $2a$ as shown in the figure. The equidistant plane (i.e. where the distances between this plane and each charge in the set are equal) is the *xy* plane.

1. Show that the T_{zz}, T_{xz}, T_{yz} components of the Maxwell stress tensor in the equidistant plane are

$T_{zz} = -\frac{q^2}{2(2\pi)^2\epsilon_0}\frac{r^2}{(a^2+r^2)^3}$; $T_{xz} = T_{yz} = 0$ (7 points)

2. Determine the force on the upper charge by integrating the Maxwell stress tensor over the equidistant plane. (7 points)

3. Explain why your result makes sense (1 point)

Tip 1: you might find useful the following integral: $\int_0^\infty \frac{r^3}{(r^2+a^2)^3} dr = \frac{1}{4a^2}$

Tip 2: you might find useful the surface element $d\vec{\mathbf{a}}$ in the *xy* plane in the cylindrical coordinates $d\vec{\mathbf{a}} = (0, 0, -r\, dr d\varphi)$

Answers to Question 2 (Problem 8.4) (15 points)

1. The electric field from one charge:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$ 1 point

In the *xy*-plane:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}2\frac{q}{\mathscr{r}^2}cos\theta\,\hat{\mathscr{r}};\; cos\theta = \frac{r}{\mathscr{r}} = \frac{r}{\sqrt{a^2+r^2}}$$ 2 points

$E_z = 0$ (also because of symmetry) 1 point

$$E^2 = \left(\frac{q}{2\pi\epsilon_0}\right)^2\frac{r^2}{(a^2+r^2)^3}$$

$$T_{zz} \equiv \epsilon_0\left(E_zE_z - \frac{1}{2}E^2\right) = \epsilon_0\left(-\frac{1}{2}\left(\frac{q}{2\pi\epsilon_0}\right)^2\frac{r^2}{(a^2+r^2)^3}\right) = -\frac{q^2}{2(2\pi)^2\epsilon_0}\frac{r^2}{(a^2+r^2)^3}$$ 2 points

$T_{xz} \equiv \epsilon_0 E_x E_z = 0;\; T_{yz} = 0$ 1 point

(7 points in total)

2.

$$\vec{\mathbf{F}} = \oint_S \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \epsilon_0\mu_0\frac{d}{dt}\int_V \vec{\mathbf{S}}\,d\tau = \oint_S \overleftrightarrow{\mathbf{T}}\cdot d\vec{\mathbf{a}}$$ 1 point

$$\overleftrightarrow{\mathbf{T}} = \begin{pmatrix} T_{xx} & T_{xy} & 0 \\ T_{yx} & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{pmatrix}$$

$d\vec{\mathbf{a}} = (0, 0, -r\,dr\,d\varphi)$ (in cylindrical coordinates)

so only the $T_{zz}da_z$ component is non-zero and therefore only F_z in non-zero

$F_x = F_y = 0$ 2 points

$$F_z = \oint_S T_{zz}da_z = \iint \frac{q^2}{2(2\pi)^2\epsilon_0}\frac{r^2}{(a^2+r^2)^3}r\,dr\,d\varphi = \frac{q^2}{2(2\pi)^2\epsilon_0}2\pi\int_0^\infty \frac{r^3}{(a^2+r^2)^3}dr = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2a)^2}$$

(7 points in total) 4 points

<u>**Note added**</u>: Some asked if there is a typo in the integral in Griffiths' solutions manual – yes, it is: the cube should be substituted with a square:

$$\int\frac{u\,du}{(u+a^2)^3} = \{x = u + a^2\} = \int\frac{(x-a^2)\,dx}{x^3} = \int\frac{dx}{x^2} - a^2\int\frac{dx}{x^3} = -\frac{1}{x} + \frac{a^2}{2}\frac{1}{x^2} = -\frac{1}{u+a^2} + \frac{a^2}{2\,(u+a^2)^{\cancel{3}2}}$$

For Question 2 it doesn't matter as the right value of the integral already provided.

3. This is exactly the force according to Coulomb's law. (1 point)

Question 3 (15 points)

A plane electromagnetic wave travelling through vacuum in the positive z direction and polarized along the x direction, encounters a perfect conductor, occupying the region $z \geq 0$, and reflects back. The electric field inside a perfect conductor is zero.

1. Show, by invoking the proper boundary condition, that the complete electric field of the plane electromagnetic wave in the $z < 0$ region is $\vec{\mathbf{E}} = E_0[cos(kz - \omega t) - cos(kz + \omega t)]\,\hat{\mathbf{x}}$

(4 points)

2. Show that the accompanying magnetic field in the $z < 0$ region is

$\vec{\mathbf{B}} = \frac{E_0}{c}[cos(kz - \omega t) + cos(kz + \omega t)]\,\hat{\mathbf{y}}$ (3 points)

3. Assuming $\vec{\mathbf{B}} = \mathbf{0}$ inside the conductor, find the current $\vec{\mathbf{K}}$ on the surface $z = 0$, by invoking the appropriate boundary condition. (3 points)

4. Find the time-averaged magnetic force $\vec{\mathbf{f}}$ per unit area on the surface (Tip: $\vec{\mathbf{f}} = \vec{\mathbf{K}} \times \vec{\mathbf{B}}$)

(3 points)

5. Calculate the expected radiation pressure and compare your result with it. (2 points)

Answer to Question 3 (Griffiths, Problem 9.34 modified)

1. (4 points)

Because the EM wave orthogonal to the interface, the boundary condition

$\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0$

$\mathbf{E}_2^{\parallel} = 0$ because the conductor is perfect (1 point)

$\mathrm{E}_\mathrm{I} + \mathrm{E}_R = 0$; $\mathrm{E}_\mathrm{R} = -\mathrm{E}_I$ - the reflected wave has a π phase shift (1 point)

$\mathbf{E} = E_0[cos(kz - \omega t) - cos(kz + \omega t)]\hat{\mathbf{x}}$ (2 points)

(-1 point if no $\hat{\mathbf{x}}$)

(-1 point if -kz)

2. (3 points)

$\mathbf{B} = \frac{E_0}{c}[cos(kz - \omega t) + cos(kz + \omega t)]\hat{\mathbf{y}}$

$B_0 = \frac{E_0}{c}$ because of scaling of the magnetic field (-1 point if incorrect)

$\hat{\mathbf{y}}$ because of polarization along $\hat{\mathbf{x}}$ and propagation along $\hat{\mathbf{z}}$ (-1 point if incorrect)

The "+" sign because **E** changes the sign upon reflection but **B** does not, and **E** × **B** is directed to the propagation direction (-1 point if incorrect)

3. (3 points)

The boundary condition $\frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$ is given in the extended formula sheet

$$\vec{\mathbf{K}} \times (-\hat{\mathbf{z}}) = \frac{1}{\mu_0}\vec{\mathbf{B}} = \frac{E_0}{\mu_0 c}[2cos(\omega t)]\,\hat{\mathbf{y}};\ \vec{\mathbf{K}} = \frac{2E_0}{\mu_0 c}cos(\omega t)\,\hat{\mathbf{x}}$$

4. (3 points)

The force per unit area at z=0 is

$$\vec{\mathbf{f}} = \vec{\mathbf{K}} \times \vec{\mathbf{B}} = \frac{2E_0^2}{\mu_0 c^2}[cos(\omega t)\,\hat{\mathbf{x}}] \times [cos(\omega t)\,\hat{\mathbf{y}}] = 2\epsilon_0 E_0^2 cos^2(\omega t)\,\hat{\mathbf{z}}\ \left\{= \frac{2E_0^2}{\mu_0 c^2}cos^2(\omega t)\,\hat{\mathbf{z}}\right\}$$

__**Note added:**__ at the first glance, there is a factor of 2 missing here (one multiplier of 2 from $\vec{\mathbf{B}}$ and another 2 from $\vec{\mathbf{K}}$). However, the magnetic field has the amplitude of $B_{z<0} = 2cos(\omega t)$ at $z < 0$ and $B_{z>0} = 0$ at $z > 0$ while the force is calculated exactly at $z = 0$. Therefore, the "effective" magnetic field applied to the electrons, is an average of the two: $B_{z=0}^{eff} = (B_{z<0} + B_{z>0})/2 = B_{z<0}/2$. For more discussion on the point, see Chapter 2.5.3 (it's about the electric field but the idea is the same). No points are deduced if the factor of 2 is still present in the answer.

The time average of $cos^2(t) = 0.5$, so $\vec{\mathbf{f}}_{ave} = \epsilon_0 E_0^2\,\hat{\mathbf{z}}\left\{= \frac{E_0^2}{\mu_0 c^2}\,\hat{\mathbf{z}}\right\}$

5. This is twice the radiation pressure $P = I/c = \frac{1}{2}\epsilon_0 E_0^2$ calculated for a perfect absorber, whereas this is a perfect reflector. (2 points)

Question 4 (5 points)

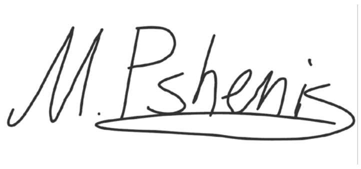
Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero that is traveling in the negative x direction and polarized in the z direction.

Answer to Question 4 (Griffiths, Problem 9.9a) (5 points)

$$\vec{\mathbf{k}} = -\frac{\omega}{c}\,\hat{\mathbf{x}};\;\; \hat{\mathbf{n}} = \hat{\mathbf{z}};\; \vec{\mathbf{k}}\cdot\vec{\mathbf{r}} = \left(-\frac{\omega}{c}A\,\hat{\mathbf{x}}\right)\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = -\frac{\omega}{c}x;\; \vec{\mathbf{k}}\times\hat{\mathbf{n}} = -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}$$

$$\vec{\mathbf{E}}(x,t) = E_0 cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{z}};\;\; \vec{\mathbf{B}}(x,t) = \frac{E_0}{c} cos\left(\frac{\omega}{c}x + \omega t\right)\hat{\mathbf{y}}$$

-1 point if -*kx* (wrong direction)
-1 point if wrong polarization
-1 point if $\vec{\mathbf{E}}$ and/or $\vec{\mathbf{B}}$ are not vectors
-1 point if $\vec{\mathbf{B}}$-direction is not correct
-1 point if B-scaling is not correct
-0.5 point if k enters the answer as k was not given.



Maxim Pchenitchnikov
May 21 2019

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